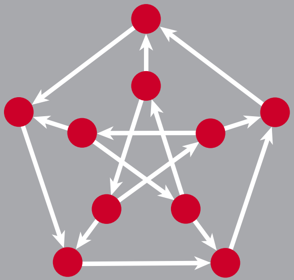

Fast Algorithms for Pseudoarboricity

Meeting on Algorithm Engineering and Experiments – ALENEX 2016

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A tree...

...an unrooted tree...

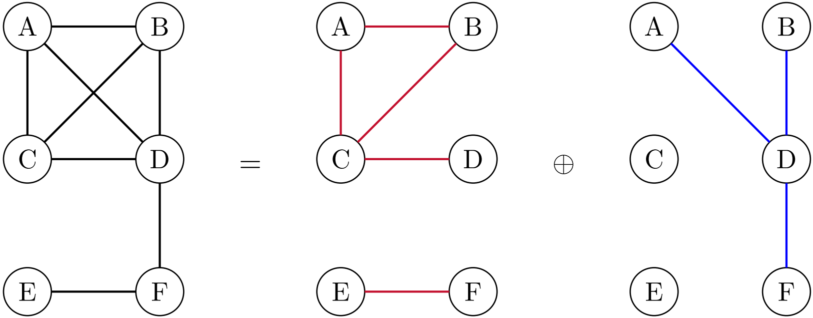
...a pseudotree!

?!

Pseudoarboricity

Definition

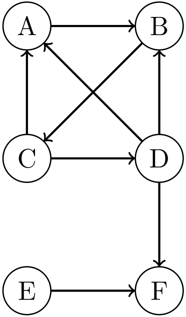
The pseudoarboricity $p(G)$ of an undirected graph G is the minimum number of pseudoforests into which the graph can be decomposed.



Pseudoarboricity and Orientations

Theorem (Frank–Gyárfás 1976 + Picard–Queyranne 1982)

Let $\vec{G}$ be an orientation of G such that the maximum indegree is minimal. Then this indegree equals the pseudoarboricity $p(G)$.



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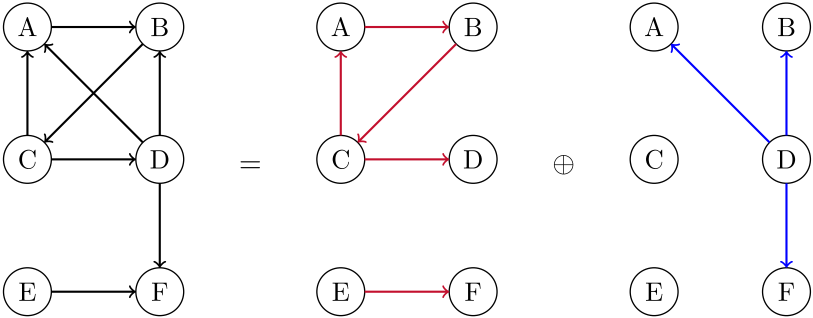

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- The test can be performed with a maximum flow algorithm, which ‘re-orients’ an arbitrary orientation to a d-orientation by reversing directed paths, if possible

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- The test can be performed with a maximum flow algorithm, which 're-orients' an arbitrary orientation to a d-orientation by reversing directed paths, if possible
- With Dinitz's algorithm, we need $\mathcal{O}(|E|^{3/2} \log |I|)$ time on a search interval I that contains p

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Approximating p

- The algorithm can be turned into an approximation scheme which returns d satisfying $p \leq d \leq \lceil (1+\epsilon)p \rceil$ in time

$$\mathcal{O}\left(|E| \, \frac{\log|V|}{\epsilon} \, \log|I|\right)$$

  for $\epsilon > 0$ on a search interval I containing p
- The approximation is not achieved by stopping the binary search early, but by stopping Dinitz's algorithm after $2 + \log_{1+\epsilon}|V|$ phases (Kowalik 2006)

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- First compute a $(1+\epsilon)$-approximation, then the exact algorithm can be performed in $\mathcal{O}(\log(\epsilon p))$ tests
- We set ϵ to balance the runtimes of the approximation and the exact phase
- We will use the fact that always $p \in \mathcal{O}(\sqrt{|E|})$ holds

A Selection of New Results

Bound on p	Runtime bound to compute p
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Comparison: matroid partitioning algorithm (Westermann 1988):

$$\mathcal{O}\left(\lvert E\rvert^{3/2}\sqrt{\log p}\right)$$

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How to obtain $\mathcal{O}\left(|E|^{3/2} \log\log|V|\right)$ unconditionally

- Use a $(1+\epsilon)$-approximation to accelerate the exact algorithm!
- Set

$$\epsilon = \frac{(\log|V|)^2}{p}.$$

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- A binary search in an exact algorithm needs $\mathcal{O}(\log(\epsilon p))$ tests on the narrowed search interval – the runtime is

$$\mathcal{O}(|E|^{3/2}\log\log^{\cancel{2}}|V|).$$

Sketch: Obtaining $\mathcal{O}\left(|E|^{3/2}\log^* p\right)$ if $p \in \mathcal{O}(\sqrt{|E|}/\log|V|)$

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$$\epsilon_1 \simeq \frac{\log p}{p},\ \epsilon_2 \simeq \frac{\log \log p}{p},\ \epsilon_3 \simeq \frac{\log \log \log p}{p},\ ...,\ \epsilon_k \simeq \frac{\log^{\times k} p}{p}$$

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- The initial interval size is $|I_0| \in \mathcal{O}(p)$ with the 2-approximation.
- In every phase $i = 1, ..., k$, we reduce the interval to size

$$|I_i| \in \mathcal{O}\left(\log^{\times i} p\right).$$

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- The total runtime is $\mathcal{O}\left(|E|^{3/2}\log^* p\right)$

What have we shown?

Bound on p	Runtime bound to compute p	
–	$\mathcal{O}\left(\lvert E\rvert^{3/2}\sqrt{\log\log p}\right)$	($\checkmark$)
$\mathcal{O}\left(\frac{\sqrt{\lvert E\rvert}}{\log\lvert V\rvert}\right)$	$\mathcal{O}\left(\lvert E\rvert^{3/2}\log^* p\right)$	$\checkmark$

Cmp. Westermann 1988: $\mathcal{O}\left(\lvert E\rvert^{3/2}\sqrt{\log p}\right)$

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What have we shown?

Bound on p	Runtime bound to compute p	
$\Omega(\sqrt{\lvert E\rvert})$	$\mathcal{O}\left(\lvert E\rvert^{3/2}\right)$	up next
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$\mathcal{O}\left(\frac{\sqrt{\lvert E\rvert}}{\log\lvert V\rvert}\right)$	$\mathcal{O}\left(\lvert E\rvert^{3/2}\log^{*} p\right)$	✓

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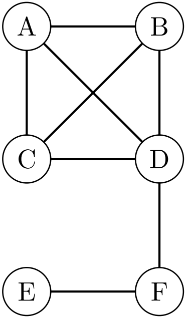
The Densest Subgraph Problem



Definition

Let $G = (V, E)$ be an undirected graph. The maximum density is defined as

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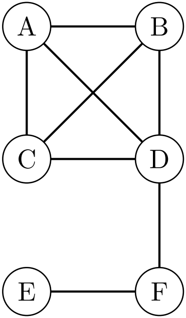

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Observation (Khuller and Saha 2009)

A vertex v with $\deg(v) < d^*(G)$ cannot be in a densest subgraph.

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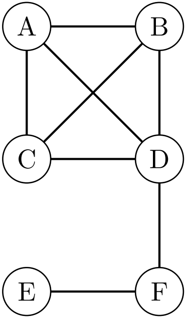

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Theorem (Picard–Queyranne 1982)

For any undirected graph G, we have $\lceil d^*(G) \rceil = p$.

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Preprocessing

Observation (Khuller and Saha 2009)

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- For the resulting graph $G' = (V', E')$, we have $d^*(G') = d^*(G)$ and thus $p(G') = p(G)$ by the Picard–Queyranne theorem

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- For the resulting graph $G' = (V', E')$, we have $d^*(G') = d^*(G)$ and thus $p(G') = p(G)$ by the Picard–Queyranne theorem
- G' is possibly smaller (great!) and has a higher average density, i.e.

$$\frac{|E'|}{|V'|} \geq \frac{|E|}{|V|}.$$

Preprocessing – A Theoretical Guarantee

- Utilize Gabow's algorithm, which runs in $\mathcal{O}\left(|E|^{3/2} \log \frac{|V|^2}{|E|}\right)$ time

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If $d^*(G) \in \Omega(\sqrt{|E|})$, then we can obtain a subgraph $G' = (V', E')$ with $|E'| \in \Theta(|V'|^2)$ and $d^*(G') = d^*(G)$ in linear time.

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Corollary

If $d^*(G) \in \Omega(\sqrt{|E|})$, p can be determined in $\mathcal{O}(|E|^{3/2})$.

The Re-orientation Algorithm in Practice (Dinitz's algorithm)

		Without preprocessing		With preprocessing	
Graph	p	$\|E\|$	Runtime [s]	$\|E'\|$	Runtime [s]
Amazon	5	900K	23	800K	1
DBLP	57	1M	4	14K	0
YouTube	46	3M	4	417K	0
LiveJournal	194	35M	251	540K	0
Orkut	228	117M	1012	13M	33

Note: Push-relabel algorithms are slower by an order of magnitude

Open Questions

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- How much are certain graph families, e.g. power-law graphs, reduced by preprocessing?
- A 2-approximation of p (and d^*) can be found in $\mathcal{O}(|E|)$ time with a greedy algorithm. Is a factor smaller than 2 possible in linear time?
  (Any constant-factor approximation can be found in $\mathcal{O}(|E| \log |V| \log p)$ time)

Thank you for your attention!